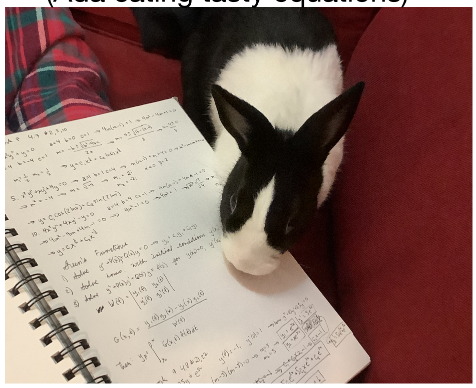
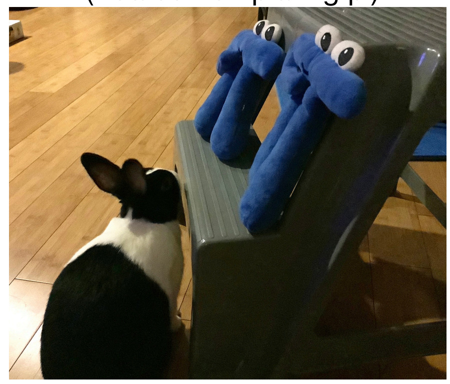
This is my bunny, Ada.

(Ada eating tasty equations)



(Ada contemplating pi)



Today we are going to experiment on Ada (safely and humanely, as you will see) by putting her into some very interesting machines, and seeing what kinds of things can happen to her.

A company has just come out with a range of "growing machines." Each machine increases the size of what's inside by a certain factor each minute.

I decide to put Ada in the doubling machine. I don't care for the moment about specific measurements so I just define her starting size as 1x starting size.

size = 1 → $\boxed{2}$ → size after:
1 min: 2
2 min: 4
3 min: 8

So in general, her size will be 2^t, with t = # of min.

How about putting her in a tripling machine?

1 → $\boxed{3}$ →
1 min: 3
2 min: 9
3 min: 27

Size $= 3^t$ ... and so on with any base of machine.

Q: Let's say I put her in the doubling machine for 4 min. I now have a bunny of size 16, and am going to bankrupt myself buying lettuce. How can I get her back to normal?

A: - I could go out and buy a "halving" machine to put her in for 4 min.

- But I don't have to: I can instead put her back in the doubling machine for negative time! $2^{-4} = \frac{1}{2^4} = \left(\frac{1}{2}\right)^4$ So I created a halving machine from my doubling machine.

- And actually, I can create *any* base of machine from any other, by adjusting the coefficient of the exponent. So really, we only need one machine... and 99% of the time, we're going to choose the base-e machine.

Q: Is this dangerous for her? Could I ever leave her in on negative time for so long she disappears?

A: No! Being a perfectly mathematical bunny, she can get infinitely smaller, but never become 0 (disappear.) Therefore, I'll also never come back to a negative bunny.

Q: What if I put her in for 0 time?

A: Nothing! $2^0 = 1$; "I put Ada in the doubling machine for 0 time, i.e. I actually didn't put her in at all, and she ended up $1\times$ her original size." this is an intuition for why $x^0 = 1$ always.

Q: I put her in the doubling machine, but then forget her. I come back and she's size 64. How long was she in there?

A: $2^t = 64$, need to find t. This is what logs are for!

$\log_2(64) = t \rightarrow t = 6$

Recap of exp and log rules...

$e^2 e^3 = e^{2+3} = e^5 \rightarrow$ "I put Ada in the e machine for 2 min, then took the size of rabbit she became and put it back in for 3 min. Might as well have just put her in for 5 min in the first place."

so in general...

$a^x a^y = a^{x+y}$ means that... $\log_a(xy) = \log_a(x) + \log_a(y)$

$\frac{a^x}{a^y} = a^{x-y}$ means that... $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

what if $x = y$ for an addition? Then...

$\log_a(x) + \log_a(x) = \log_a(x \cdot x) \rightarrow 2\log_a(x) = \log_a(x)^2$

And that works for any number of copies being added together:

$n\log_a(x) = \log_a(x)^n$

A bacteria culture grows exponentially. The number of cells (in millions) after t hours is given by the function $N(t) = 3e^{.07t}$.

a) What is the population after 6 hours? After 12 hours?

b) The continuous growth rate is given as 0.07. By what percentage does the bacteria culture grow every hour?

a) $N(6) = 3e^{0.07(6)} \rightarrow N(6) = 4.57$ million

$N(12) = 3e^{0.07(12)} \rightarrow N(12) = 6.95$ million

The growth is getting faster!

b) $N(0) = 3e^{0.07(0)} \rightarrow N(0) = 3$ million

$N(1) = 3e^{0.07(1)} \rightarrow N(1) = 3.2175$ million

# of new cells in 1 hour = 0.2175 million

$\frac{0.2175}{3} = 0.0725 \cdot 100 \rightarrow 7.25\%$

Another way to see this would be to convert our base -e to a "machine" like our bunny growth machines: with only t in the exponent, and the base showing the type of growth:

$$3e^{0.07t} \rightarrow 3\left(e^{0.07}\right)^t \rightarrow 3(1.0725)^t$$

So this is a growing machine that multiplies the size of whatever's inside by 1.0725 every unit of time.

To state that in percentages...

- At the beginning of each unit of time, the object is 100% of its starting size for that unit of time.

- At the end of each unit of time, the object is 107.25% its starting size.

- Therefore, it has grown by 7.25%.

Simplify the expression by writing it as a single exponential/logarithm:

a) $\dfrac{e^x e^{x^2}}{e^{2x} e}$

b) $\ln(x^2+1) - \frac{1}{2}\ln(x-1)$

a) $\dfrac{e^x e^{x^2}}{e^{2x} e^1} \rightarrow \dfrac{e^{x+x^2}}{e^{2x+1}} \rightarrow e^{(x+x^2)-(2x+1)}$

$\rightarrow e^{x+x^2-2x-1} \rightarrow e^{x^2-x-1}$

b) $\ln(x^2+1) - \ln(x-1)^{\frac{1}{2}} \rightarrow \ln\left(\dfrac{x^2+1}{\sqrt{x-1}}\right)$

Solve for x:

a) $10^x = 50$

b) $3e^x + 1 = 2$

c) $3e^{2x-1} = 1$

a) $10^x = 50 \rightarrow \log_{10}(50) = x \rightarrow x \approx 1.699$

b) $3e^x + 1 = 2 \rightarrow 3e^x = 1 \rightarrow e^x = \frac{1}{3} \rightarrow \ln(e^x) = \ln(\frac{1}{3})$

$\rightarrow x = \ln(\frac{1}{3}) \rightarrow x \approx -1.0986$

c) $3e^{2x-1} = 1 \rightarrow e^{2x-1} = \frac{1}{3} \rightarrow \ln(e^{2x-1}) = \ln(\frac{1}{3})$

$2x-1 = \ln(\frac{1}{3}) \rightarrow 2x = \ln(\frac{1}{3})+1 \rightarrow x = \dfrac{\ln(\frac{1}{3})+1}{2}$

$x \approx -0.0493$

Solve for x:

a) $\ln(x) = 10$ b) $3\log_{10}(x) - 1 = 4$ c) $\ln(x+1) + \ln(x-1) = 1$

a) $e^{\ln(x)} = e^{10} \rightarrow x = e^{10} \rightarrow x \approx 22026.46$

b) $3\log_{10}(x) - 1 = 4 \rightarrow 3\log_{10}(x) = 5 \rightarrow \log_{10}(x) = \frac{5}{3}$

$10^{\log_{10}(x)} = 10^{\frac{5}{3}} \rightarrow x = 10^{\frac{5}{3}} \rightarrow x \approx 46.42$

c) $\ln(x+1) + \ln(x-1) = 1 \rightarrow \ln[(x+1)(x-1)] = 1$

$\rightarrow \ln[x^2 - 1] = 1 \rightarrow e^{\ln(x^2-1)} = e^1$

$\rightarrow x^2 - 1 = e \rightarrow x^2 = e + 1 \rightarrow x = \pm\sqrt{e+1}$

But remember... it is impossible for me to get 0 bunny, or a negative bunny, from putting a bunny in a growing/shrinking machine.

That means it makes no sense to plug 0 or a negative number into a log function. Stated otherwise: the domain of $f(x) = \log_b(x)$ is $x > 0$

therefore

$$x = \pm\sqrt{e+1} \;\begin{cases} \rightarrow 1.928 \\ \rightarrow -1.928 \end{cases}$$

1.928 gives inputs that are within the domain, but -1.928 does not. Therefore, $+1.928$ is the only answer.

P.S... I actually have two bunnies... here are Ada and Mips