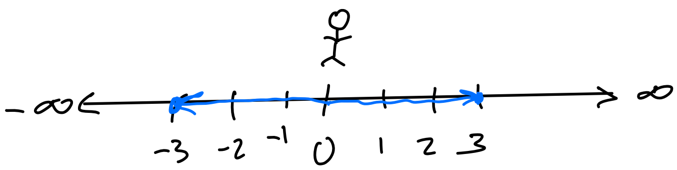
Four ways of thinking about $|x|$

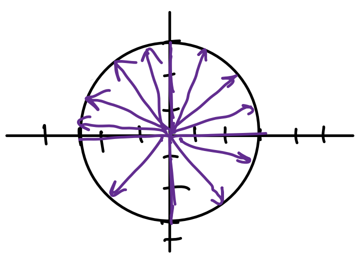
① Absolute value is the number's distance from 0



If you live at 0, and you want to go for a walk three units away from your house, you have two choices of destination: 3 or -3.

another word for "absolute value": "magnitude"

spoilers! what if, instead of just walking positive or negative, we added another number line, and you could walk any direction you wanted?



Now you have lots of choices of places to go with an "absolute value"/"magnitude" of 3... infinite choices, actually... and the collection of all of them makes a circle.

② $$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

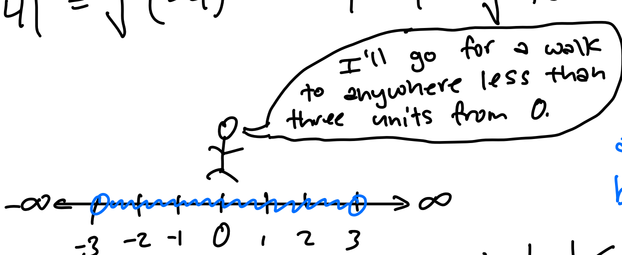
③ $$|x| = \sqrt{x^2}$$ *"Principal" square root*

Let's check that it works:

$|4| = \sqrt{4^2} \rightarrow |4| = \sqrt{16} \rightarrow |4| = 4$ ✓

$|-4| = \sqrt{(-4)^2} \rightarrow |-4| = \sqrt{16} \rightarrow |-4| = 4$ ✓

④ $$|x| < 3$$



Valid walk destinations: anywhere bigger than -3, but smaller than 3.

$\therefore |x| < 3 = -3 < x < 3$

Solve the double inequality $-3 < 6 - 3x \leq 7$

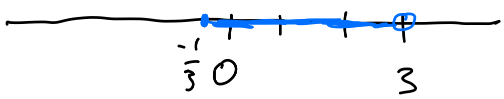
$$-3 < 6 - 3x \leq 7 \rightarrow -3 - 6 < -3x \leq 7 - 6$$

$$\rightarrow \frac{-9}{-3} < \frac{-3x}{-3} \leq \frac{1}{-3} \rightarrow 3 > x \geq -\frac{1}{3}$$

$$\rightarrow -\frac{1}{3} \leq x < 3$$

$$\left[-\tfrac{1}{3}, 3\right)$$

valid values of x in here



(there are an infinite number of valid values in there!)

Recall:

[ ] endpoint included

( ) not included

Infinity is a CONCEPT, not a number, so it is NEVER INCLUDED! So, I remember which bracket is which with the fact that the curvy bracket always goes with the curvy infinity sign: $(-\infty, \infty)$

Solve the equation and verify your answers:

$$x - |2x+1| = 5$$

$$|2x+1| = \begin{cases} (2x+1) & \text{if } \underline{2x+1 \geq 0} \rightarrow 2x \geq -1 \rightarrow \boxed{x \geq -\frac{1}{2}} \\ -(2x+1) & \text{if } \underline{2x+1 < 0} \rightarrow \text{i.e if } \boxed{x < -\frac{1}{2}} \end{cases}$$

Case 1: $x \geq -\frac{1}{2}$ then $|2x+1| = (2x+1)$

$\rightarrow x - (2x+1) = 5 \rightarrow x - 2x - 1 = 5$

$\rightarrow -x = 6 \rightarrow \boxed{x = -6}$ ✗

check: $-6 - |2(-6)+1| = 5$

$-6 - |-11| = 5 \rightarrow -6 - 11 = 5 \quad -17 \neq 5$

case 2: $x < -\frac{1}{2}$ then $|2x+1| = \underline{-(2x+1)} = (-2x-1)$

$x - |2x+1| = 5 \rightarrow x - (-2x-1) = 5$

$\rightarrow x + 2x + 1 = 5 \rightarrow 3x = 4 \rightarrow \boxed{x = \frac{4}{3}}$ ✗

check: $\frac{4}{3} - \left|2\left(\frac{4}{3}\right) + 1\right| = 5 \rightarrow \frac{4}{3} - \left|\frac{8}{3} + \frac{3}{3}\right| = 5$

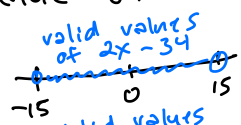
$\rightarrow \frac{4}{3} - \left|\frac{11}{3}\right| = 5 \rightarrow \frac{4}{3} - \frac{11}{3} = 5 \rightarrow -\frac{7}{3} \neq 5$

There are no valid solutions.

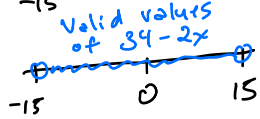
Twice the value of x is within 15 units of 34.

a) Write this sentence as an inequality.
b) Solve the inequality to find the possible range of x.

Well, $2x-34<15$ would work as long as $2x$ were bigger than 34.
and $34-2x<15$ would work as long as 34 were bigger than $2x$.
But slap an absolute value on the LHS... and they both work!

$|2x-34|<15 \longrightarrow$ 

$\longrightarrow -15<2x-34<15$

$|34-2x|<15 \longrightarrow$ 

$\longrightarrow -15<34-2x<15$

Let's solve both, and reassure ourselves that we get the same answer:

$$-15<2x-34<15 \rightarrow -15+34<2x<15+34$$

$$\rightarrow \frac{19}{2}<\frac{2x}{2}<\frac{49}{2} \rightarrow \frac{19}{2}<x<\frac{49}{2} \qquad \boxed{\left(\frac{19}{2},\frac{49}{2}\right)}$$

OR:

$$-15<34-2x<15 \rightarrow -15-34<-2x<15-34$$

$$\rightarrow \frac{-49}{-2}<\frac{-2x}{-2}<\frac{-19}{-2} \rightarrow \frac{49}{2}>x>\frac{19}{2}$$

*

$$\frac{19}{2}<x<\frac{49}{2} \qquad \boxed{\left(\frac{19}{2},\frac{49}{2}\right)}$$

*we can see from this example why the inequality signs need to be flipped when dividing by a negative number!

Solve the quadratic inequality $x^2 + 10x - 2 < 0$
by first completing the square on the left hand side.

$$x^2 + 10x - 2 < 0$$

$$\hookrightarrow \div 2 = \boxed{5} \quad 5^2 = 25$$

$$\underbrace{x^2 + 10x + 25}\ \underbrace{-2 - 25} < 0 \rightarrow (x+5)(x+5) - 27 < 0$$

$$\rightarrow (x+5)^2 < 27$$

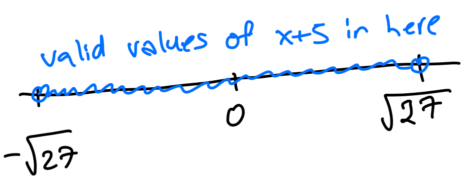
and recall: $|x| = \sqrt{x^2}$

so therefore, $\sqrt{x^2} = |x|$

and therefore, $\sqrt{(x+5)^2} = |x+5|$

so to get rid of the exponent, take the root on both sides:

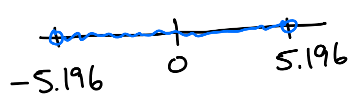
$$\sqrt{(x+5)^2} < \sqrt{27} \longrightarrow |x+5| < \sqrt{27}$$



$$-\sqrt{27} < x+5 < \sqrt{27} \longrightarrow \boxed{-\sqrt{27} - 5 < x < \sqrt{27} - 5}$$

Could also use the approximate decimal value of $\sqrt{27}$

$\sqrt{27} \approx 5.196$



$$-5.196 < x+5 < 5.196 \longrightarrow -10.196 < x < 0.196$$

Or express $\sqrt{27}$ simplified : $\sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3}$